The Transmission of Electric Waves Around the Earth's Surface.

By H. M. Macdonald, F.R.S., Professor of Mathematics in the University of Aberdeen.

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The value of the magnetic force at a point on the earth's surface, due to a simple oscillator placed on the surface with its axis normal to the surface, has been recently calculated by Love* for a wave-length of 5 kilom. at certain distances from the oscillator. His results for the case of perfect conduction are the same as those obtained by the writer.† He also calculated the value of the sum of the corresponding series when the surface of the earth is supposed to be imperfectly conducting. The object of the present communication is to obtain the general formula for the case of imperfect conduction.

Let r, θ, ϕ be the polar co-ordinates of a point, where r is its distance from the centre of the earth, θ its angular distance from the oscillator, E_r, E_θ, E_ϕ the components of the electric force, and α, β, γ the corresponding components of the magnetic force. Then, since there is symmetry round the axis of the oscillator,

$$\alpha = 0, \qquad \beta = 0, \qquad E_\phi = 0;$$

and throughout space outside the surface

$$\left.\begin{aligned} \frac{1}{V^2}\frac{\partial E_r}{\partial t} &= \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}(\gamma r \sin\theta), \\ \frac{1}{V^2}\frac{\partial E_\theta}{\partial t} &= -\frac{1}{r\sin\theta}\frac{\partial}{\partial r}(\gamma r\sin\theta), \\ -\frac{\partial\gamma}{\partial t} &= \frac{1}{r}\frac{\partial}{\partial r}(rE_\theta) - \frac{1}{r}\frac{\partial E_r}{\partial\theta}, \end{aligned}\right\} \tag{1}$$

where V is the velocity of propagation of the waves. Eliminating E_r, E_θ, it follows that γ satisfies the differential equation

$$\frac{1}{V^2}\frac{\partial^2}{\partial t^2}(\gamma r\sin\theta) = \frac{\partial^2}{\partial r^2}(\gamma r \sin\theta) + \frac{1}{r^2}(1-\mu^2)\frac{\partial^2}{\partial\mu^2}(\gamma r\sin\theta), \tag{2}$$

where $\mu = \cos\theta$, throughout the external space. Further, if σ is the specific resistance of the conducting material, the equations to be satisfied throughout the internal space are

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* 'Phil. Trans.,' A, vol. 215, p. 105 (1915).

† 'Roy. Soc. Proc.,' vol. 90, p. 50 (1914).

$$\left.\begin{aligned} 4\pi \mathrm{E}_r/\sigma &= \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}(\gamma r \sin\theta), \\ 4\pi \mathrm{E}_\theta/\sigma &= -\frac{1}{r\sin\theta}\frac{\partial}{\partial r}(\gamma r \sin\theta), \\ -\frac{\partial\gamma}{\partial t} &= \frac{1}{r}\frac{\partial}{\partial r}(r\mathrm{E}_\theta) - \frac{1}{r}\frac{\partial \mathrm{E}_r}{\partial\theta}, \end{aligned}\right\} \tag{3}$$

and therefore, eliminating E_r, E_θ, γ satisfies the differential equation

$$\frac{4\pi}{\sigma}\frac{\partial}{\partial t}(\gamma r\sin\theta) = \frac{\partial^2}{\partial r^2}(\gamma r\sin\theta) + \frac{1}{r^2}(1-\mu^2)\frac{\partial^2}{\partial\mu^2}(\gamma r\sin\theta), \tag{4}$$

throughout the internal space; the boundary conditions at the surface are that the tangential components of the electric and magnetic forces are continuous. For waves of wave-length $2\pi/\kappa$, writing

$$\gamma r\sin\theta = \psi e^{\iota\kappa \mathrm{V}t}$$

in the external space, and

$$\gamma r\sin\theta = \psi' e^{\iota\kappa \mathrm{V}t}$$

in the internal space, the equations to be satisfied by ψ and ψ' are

$$\frac{d^2\psi}{\partial r^2} + \frac{1}{r^2}(1-\mu^2)\frac{\partial^2\psi}{\partial\mu^2} + \kappa^2\psi = 0, \qquad r > a, \tag{5}$$

$$\frac{\partial^2\psi'}{\partial r^2} + \frac{1}{r^2}(1-\mu^2)\frac{\partial^2\psi'}{\partial\mu^2} - \frac{4\pi\iota\kappa \mathrm{V}}{\sigma}\psi', \qquad r < a, \tag{6}$$

$$\frac{\sigma}{4\pi r\sin\theta}\frac{\partial\psi'}{\partial r} = \frac{\mathrm{V}}{\iota\kappa r\sin\theta}\frac{\partial\psi}{\partial r}, \qquad \psi' = \psi, \quad r = a, \tag{7}$$

where a is the radius of the earth. If ψ_1 is the value of ψ due to the oscillation alone, it is known that

$$\psi_1 = -r_1^{-\frac{3}{2}} r^{\frac{1}{2}} \sum_1^\infty (2n+1) e^{\frac{1}{2}(n+\frac{1}{2})\pi\iota} \mathrm{K}_{n+\frac{1}{2}}(\iota\kappa r_1) \mathrm{J}_{n+\frac{1}{2}}(\kappa r)(1-\mu^2)\frac{\partial \mathrm{P}_n}{\partial\mu}, {}^{*} \tag{8}$$

where r_1 is the distance of the oscillator from the centre, hence the solutions of (5) and (6) may be written

$$\begin{aligned} \psi = -r_1^{-\frac{3}{2}} r^{\frac{1}{2}} \sum_1^\infty (2n+1)\, & e^{\frac{1}{2}(n+\frac{1}{2})\pi\iota} \mathrm{K}_{n+\frac{1}{2}}(\iota\kappa r_1) \\ & \times \{\mathrm{J}_{n+\frac{1}{2}}(\kappa r) + \mathrm{A}_n \mathrm{K}_{n+\frac{1}{2}}(\iota\kappa r)\}(1-\mu^2)\frac{\partial \mathrm{P}_n}{\partial\mu}, \end{aligned} \tag{9}$$

$$\psi' = r^{\frac{1}{2}} \sum_1^\infty \mathrm{B}_n \mathrm{J}_{n+\frac{1}{2}}(\kappa' r)(1-\mu^2)\frac{\partial \mathrm{P}_n}{\partial\mu}, \tag{10}$$

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[*] 'Roy. Soc. Proc.,' vol. 71, p. 251 (1903).

where $$\kappa'^2 = -4\pi\iota\kappa \mathrm{V}/\sigma, \tag{11}$$

and the surface conditions are

$$\frac{\sigma}{4\pi}\sum_1^\infty \mathrm{B}_n \frac{\partial}{\partial a}\{a^{\frac12}\mathrm{J}_{n+\frac12}(\kappa' a)\}\,(1-\mu^2)\frac{\partial \mathrm{P}_n}{\partial \mu}$$

$$= -\frac{\mathrm{V}}{\iota\kappa} r_1^{-\frac32}\sum_1^\infty (2n+1)e^{\frac12(n+\frac12)\pi\iota}\mathrm{K}_{n+\frac12}(\iota\kappa r_1)$$

$$\times\left[\frac{\partial}{\partial a}\{a^{\frac12}\mathrm{J}_{n+\frac12}(\kappa a)\}+\mathrm{A}_n\frac{\partial}{\partial a}\{a^{\frac12}\mathrm{K}_{n+\frac12}(\iota\kappa a)\}\right](1-\mu^2)\frac{\partial \mathrm{P}_n}{\partial \mu},$$

$$\sum_1^\infty \mathrm{B}_n\mathrm{J}_{n+\frac12}(\kappa' a)(1-\mu^2)\frac{\partial \mathrm{P}_n}{\partial \mu} = -r_1^{-\frac32}\sum_1^\infty (2n+1)e^{\frac12(n+\frac12)\pi\iota}\mathrm{K}_{n+\frac12}(\iota\kappa r_1)$$

$$\times\left[\mathrm{J}_{n+\frac12}(\kappa a)+\mathrm{A}_n\mathrm{K}_{n+\frac12}(\iota\kappa a)\right](1-\mu^2)\frac{\partial \mathrm{P}_n}{\partial \mu},$$

whence

$$-\frac{\mathrm{V}}{\iota\kappa} r_1^{-\frac32}(2n+1)e^{\frac12(n+\frac12)\pi\iota}\mathrm{K}_{n+\frac12}(\iota\kappa r_1)\left[\frac{\partial}{\partial a}\{a^{\frac12}\mathrm{J}_{n+\frac12}(\kappa a)\}+\mathrm{A}_n\frac{\partial}{\partial a}\{a^{\frac12}\mathrm{K}_{n+\frac12}(\iota\kappa a)\}\right]$$

$$= \frac{\sigma}{4\pi}\mathrm{B}_n\frac{\partial}{\partial a}\{a^{\frac12}\mathrm{J}_{n+\frac12}(\kappa' a)\},$$

$$-r_1^{-\frac32}(2n+1)e^{\frac12(n+\frac12)\pi\iota}\mathrm{K}_{n+\frac12}(\iota\kappa r_1)\left[\mathrm{J}_{n+\frac12}(\kappa a)+\mathrm{A}_n\mathrm{K}_{n+\frac12}(\iota\kappa a)\right] = \mathrm{B}_n\mathrm{J}_{n+\frac12}(\kappa a),$$

and eliminating B_n it follows that

$$\frac{\mathrm{V}}{\iota\kappa}\mathrm{J}_{n+\frac12}(\kappa' a)\left[\frac{\partial}{\partial a}\{a^{\frac12}\mathrm{J}_{n+\frac12}(\kappa a)\}+\mathrm{A}_n\frac{\partial}{\partial a}\{a^{\frac12}\mathrm{K}_{n+\frac12}(\iota\kappa a)\}\right]$$

$$= \frac{\sigma}{4\pi}\frac{\partial}{\partial a}\{a^{\frac12}\mathrm{J}_{n+\frac12}(\kappa' a)\}\left[\mathrm{J}_{n+\frac12}(\kappa a)+\mathrm{A}_n\mathrm{K}_{n+\frac12}(\iota\kappa a)\right],$$

that is

$$\mathrm{A}_n\left[\frac{4\pi\mathrm{V}}{\iota\kappa\sigma}\mathrm{J}_{n+\frac12}(\kappa' a)\frac{\partial}{\partial a}\{a^{\frac12}\mathrm{K}_{n+\frac12}(\iota\kappa a)\}-\mathrm{K}_{n+\frac12}(\iota\kappa a)\frac{\partial}{\partial a}\{a^{\frac12}\mathrm{J}_{n+\frac12}(\kappa' a)\}\right]$$

$$= \mathrm{J}_{n+\frac12}(\kappa a)\frac{\partial}{\partial a}\{a^{\frac12}\mathrm{J}_{n+\frac12}(\kappa' a)\}-\frac{4\pi\mathrm{V}}{\iota\kappa\sigma}\mathrm{J}_{n+\frac12}(\kappa' a)\frac{\partial}{\partial a}\{a^{\frac12}\mathrm{J}_{n+\frac12}(\kappa a)\}. \tag{12}$$

Therefore in the external space

$$\psi = -r_1^{-\frac32}r^{\frac12}\sum_1^\infty (2n+1)e^{\frac12(n+\frac12)\pi\iota}\mathrm{K}_{n+\frac12}(\iota\kappa r_1)$$

$$\times\left[\mathrm{J}_{n+\frac12}(\kappa r)+\frac{\mathrm{J}_{n+\frac12}(\kappa a)\frac{\partial}{\partial a}\{a^{\frac12}\mathrm{J}_{n+\frac12}(\kappa' a)\}-\frac{4\pi\mathrm{V}}{\iota\kappa\sigma}\mathrm{J}_{n+\frac12}(\kappa' a)\frac{\partial}{\partial a}\{a^{\frac12}\mathrm{J}_{n+\frac12}(\kappa a)\}}{\frac{4\pi\mathrm{V}}{\iota\kappa\sigma}\mathrm{J}_{n+\frac12}(\kappa' a)\frac{\partial}{\partial a}\{a^{\frac12}\mathrm{K}_{n+\frac12}(\iota\kappa a)\}-\mathrm{K}_{n+\frac12}(\iota\kappa a)\frac{\partial}{\partial a}\{a^{\frac12}\mathrm{J}_{n+\frac12}(\kappa' a)\}}\right.$$

$$\left.\times\,\mathrm{K}_{n+\frac12}(\iota\kappa r)\right](1-\mu^2)\frac{\partial \mathrm{P}_n}{\partial \mu}. \tag{13}$$

At the surface, $r = a$, this becomes

$$\psi = -r_1^{-\frac{3}{2}} a^{\frac{1}{2}} \sum_1^\infty (2n+1)\, e^{\frac{1}{2}(n+\frac{1}{2})\pi\iota} \mathrm{K}_{n+\frac{1}{2}}(\iota\kappa r_1)$$

$$\times \frac{\dfrac{4\pi\mathrm{V}}{\iota\kappa\sigma} \mathrm{J}_{n+\frac{1}{2}}(\kappa' a) \left[\mathrm{J}_{n+\frac{1}{2}}(\kappa a) \dfrac{\partial}{\partial a} \{a^{\frac{1}{2}} \mathrm{K}_{n+\frac{1}{2}}(\iota\kappa a)\} - \mathrm{K}_{n+\frac{1}{2}}(\iota\kappa a) \dfrac{\partial}{\partial a} \{a^{\frac{1}{2}} \mathrm{J}_{n+\frac{1}{2}}(\kappa a)\} \right]}{\dfrac{4\pi\mathrm{V}}{\iota\kappa\sigma} \mathrm{J}_{n+\frac{1}{2}}(\kappa' a) \dfrac{\partial}{\partial a} \{a^{\frac{1}{2}} \mathrm{K}_{n+\frac{1}{2}}(\kappa a)\} - \mathrm{K}_{n+\frac{1}{2}}(\iota\kappa a) \dfrac{\partial}{\partial a} \{a^{\frac{1}{2}} \mathrm{J}_{n+\frac{1}{2}}(\kappa' a)\}}$$

$$\times (1-\mu^2) \frac{\partial \mathrm{P}_n}{\partial \mu}. \tag{14}$$

Now

$$\mathrm{J}_{n+\frac{1}{2}}(\kappa a) \frac{\partial}{\partial a} \{a^{\frac{1}{2}} \mathrm{K}_{n+\frac{1}{2}}(\iota\kappa a)\} - \mathrm{K}_{n+\frac{1}{2}}(\iota\kappa a) \frac{\partial}{\partial a} \{a^{\frac{1}{2}} \mathrm{J}_{n+\frac{1}{2}}(\kappa a)\} = -a^{-\frac{1}{2}} e^{-\frac{1}{2}(n+\frac{1}{2})\pi\iota},$$

therefore

$$\psi = r_1^{-\frac{3}{2}} \sum_1^\infty \frac{(2n+1)\, \mathrm{K}_{n+\frac{1}{2}}(\iota\kappa r_1)}{\dfrac{\partial}{\partial a} \{a^{\frac{1}{2}} \mathrm{K}_{n+\frac{1}{2}}(\iota\kappa a)\} - \dfrac{\iota\kappa\sigma}{4\pi\mathrm{V}} \mathrm{K}_{n+\frac{1}{2}}(\iota\kappa a) \dfrac{\partial}{\partial a} \{a^{\frac{1}{2}} \mathrm{J}_{n+\frac{1}{2}}(\kappa' a)\} \Big/ \mathrm{J}_{n+\frac{1}{2}}(\kappa' a)}$$

$$\times (1-\mu^2) \frac{\partial \mathrm{P}_n}{\partial \mu}.$$

The important terms of this series* are those for which $|n+\frac{1}{2}-\kappa a|$ is of the same order or of lower order than $(\kappa a)^{\frac{1}{3}}$, therefore, if $|\kappa' a|$ is large compared with κa a sufficient approximation for $\mathrm{J}_{n+\frac{1}{2}}(\kappa' a)$ is given by

$$\mathrm{J}_{n+\frac{1}{2}}(\kappa' a) = (2/\pi\kappa' a)^{\frac{1}{2}} \cos \{\tfrac{1}{2}(n+1)\pi - \kappa' a\},$$

but
$$\kappa'^2/\kappa^2 = -4\pi\iota\mathrm{V}/\kappa\sigma = -2\iota\mathrm{V}\lambda/\sigma,$$

where λ is the wave-length, hence, if $\mathrm{V}\lambda/\sigma$ is a large quantity,

$$\frac{\partial}{\partial a} \{a^{\frac{1}{2}} \mathrm{J}_{n+\frac{1}{2}}(\kappa' a)\} / a^{\frac{1}{2}} \mathrm{J}_{n+\frac{1}{2}}(\kappa' a) = \kappa' \tan \{\tfrac{1}{2}(n+1)\pi - \kappa' a\},$$

that is, writing

$$\kappa' = \kappa_1(1-\iota),$$

$$\frac{\partial}{\partial a} \{a^{\frac{1}{2}} \mathrm{J}_{n+\frac{1}{2}}(\kappa' a)\} / a^{\frac{1}{2}} \mathrm{J}_{n+\frac{1}{2}}(\kappa' a) = \kappa' \frac{\tan \{\frac{1}{2}(n+1)\pi - \kappa_1 a\} + \iota \tanh \kappa_1 a}{1 - \iota \tanh \kappa_1 a \tan \{\frac{1}{2}(n+1)\pi - \kappa_1 a\}},$$

whence, approximately, since $\kappa_1 a$ is very large,

$$\frac{\partial}{\partial a} \{a^{\frac{1}{2}} \mathrm{J}_{n+\frac{1}{2}}(\kappa' a)\} / a^{\frac{1}{2}} \mathrm{J}_{n+\frac{1}{2}}(\kappa' a) = \iota\kappa',$$

when $\mathrm{V}\lambda/\sigma$ is a large quantity, and therefore

$$\psi = r_1^{-\frac{3}{2}} \sum_1^\infty \frac{(2n+1)\, \mathrm{K}_{n+\frac{1}{2}}(\iota\kappa r_1)}{\dfrac{\partial}{\partial a} \{a^{\frac{1}{2}} \mathrm{K}_{n+\frac{1}{2}}(\kappa a)\} + \dfrac{\kappa^2}{\iota\kappa'} a^{\frac{1}{2}} \mathrm{K}_{n+\frac{1}{2}}(\iota\kappa a)} (1-\mu^2) \frac{\partial \mathrm{P}_n}{\partial \mu}. \tag{15}$$

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* 'Roy. Soc. Proc.,' vol. 90, p. 50 (1914).

When the oscillator is close to the surface, r_1 can be replaced by a except for points near to the oscillator, therefore, writing

$$\kappa a = z,$$

$$\psi = \kappa z^{-2} \sum_1^\infty \frac{(2n+1)\, z^{\frac{1}{2}} \mathrm{K}_{n+\frac{1}{2}}(\iota z)}{\dfrac{\partial}{\partial z}\{z^{\frac{1}{2}}\mathrm{K}_{n+\frac{1}{2}}(\iota z)\} + \dfrac{\kappa}{\iota\kappa'}\, z^{\frac{1}{2}}\mathrm{K}_{n+\frac{1}{2}}(\iota z)} (1-\mu^2) \frac{\partial \mathrm{P}_n}{\partial \mu},$$

and for the important terms,*

$$z^{\frac{1}{2}}\mathrm{K}_{n+\frac{1}{2}}(\iota z) = \frac{2}{\pi}\, 3^{-\frac{1}{6}} z^{\frac{1}{6}} e^{-\frac{1}{2}(n+\frac{3}{2})\pi\iota} \xi^{\frac{1}{3}} \mathrm{K}_{-\frac{1}{3}}(\xi e^{-\pi\iota}),$$

$$\frac{\partial}{\partial z}\{z^{\frac{1}{2}}\mathrm{K}_{n+\frac{1}{2}}(\iota z)\} = -\frac{2}{\pi}\, 3^{\frac{1}{6}} z^{-\frac{1}{6}} e^{-\frac{1}{2}(n+\frac{3}{2})\pi\iota} \xi^{\frac{2}{3}} \mathrm{K}_{\frac{2}{3}}(\xi e^{-\pi\iota}),$$

where

$$\xi = 2\zeta^{\frac{3}{2}}, \qquad 3\zeta = 6^{\frac{1}{3}} z^{-\frac{1}{3}} (n+\tfrac{1}{2}-z),$$

whence

$$\psi = -\kappa z^{-2} \sum \frac{(2n+1)\, 3^{-\frac{1}{3}} z^{\frac{1}{3}} \xi^{\frac{1}{3}} \mathrm{K}_{\frac{1}{3}}(\xi e^{-\pi\iota})}{\xi^{\frac{2}{3}}\mathrm{K}_{\frac{2}{3}}(\xi e^{-\pi\iota}) - \dfrac{\kappa}{\iota\kappa'}\, 3^{-\frac{1}{3}} z^{\frac{1}{3}} \xi^{\frac{1}{3}} \mathrm{K}_{\frac{1}{3}}(\xi e^{-\pi\iota})} (1-\mu^2) \frac{\partial \mathrm{P}_n}{\partial \mu}, \qquad (16)$$

where the summation is extended on both sides of $n+\frac{1}{2} = z$. In order to effect the summation a knowledge of the zeros of the expression

$$\xi^{\frac{2}{3}}\mathrm{K}_{\frac{2}{3}}(\xi e^{-\pi\iota}) - \frac{\kappa}{\iota\kappa'}\, 3^{-\frac{1}{3}} z^{-\frac{1}{3}} \xi^{\frac{1}{3}} \mathrm{K}_{\frac{1}{3}}(\xi e^{-\pi\iota}),$$

is required; this expression is an integral function of $\xi^{\frac{2}{3}}$, and its zeros are continuous with the zeros of $\xi^{\frac{2}{3}}\mathrm{K}_{\frac{2}{3}}(\xi e^{-\pi\iota})$, which have been discussed in the former communication, the kth zero of $\xi^{\frac{2}{3}}\mathrm{K}_{\frac{2}{3}}(\xi e^{-\pi\iota})$ is given by $\xi^{\frac{2}{3}} = x_k^{\frac{2}{3}} e^{5\pi\iota/3}$, therefore the kth zero of

$$\xi^{\frac{2}{3}}\mathrm{K}_{\frac{2}{3}}(\xi e^{-\pi\iota}) - \frac{\kappa}{\iota\kappa'}\, 3^{-\frac{1}{3}} z^{\frac{1}{3}} \xi^{\frac{1}{3}} \mathrm{K}_{\frac{1}{3}}(\xi e^{-\pi\iota}),$$

$$\xi^{\frac{2}{3}} = x'{}_k^{\frac{2}{3}} e^{5\pi\iota/3},$$

it follows that $$x'{}_k^{\frac{2}{3}} e^{5\pi\iota/3} = x_k^{\frac{2}{3}} e^{5\pi\iota/3} + y_k,$$

and, if $|\kappa z^{\frac{1}{3}}/\kappa'|$ is a small quantity, y_k can be approximated to as a series of powers of $\kappa z^{\frac{1}{3}}/\kappa'$. Retaining the second power of $\kappa z^{\frac{1}{3}}/\kappa'$, y_k is determined by the relation

$$\frac{3}{2} y_k \xi^{\frac{1}{3}} \frac{\partial}{\partial \xi}\{\xi^{\frac{2}{3}}\mathrm{K}_{\frac{2}{3}}(\xi e^{-\pi\iota})\} + \frac{9}{8} y_k^2 \xi^{\frac{1}{3}} \frac{\partial}{\partial \xi}\left[\xi^{\frac{1}{3}} \frac{\partial}{\partial \xi}\{\xi^{\frac{2}{3}}\mathrm{K}_{\frac{2}{3}}(\xi e^{-\pi\iota})\}\right]$$
$$- \frac{\kappa}{\iota\kappa'}\, 3^{-\frac{1}{3}} z^{\frac{1}{3}} \left[\xi^{\frac{1}{3}}\mathrm{K}_{\frac{1}{3}}(\xi e^{-\pi\iota}) + \frac{3}{2} y_k \xi^{\frac{1}{3}} \frac{\partial}{\partial \xi}\{\xi^{\frac{1}{3}}\mathrm{K}_{\frac{1}{3}}(\xi e^{-\pi\iota})\}\right] = 0,$$

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* 'Roy. Soc. Proc.,' vol. 90, 1914.

where
$$\xi^{\frac{2}{3}} = x_k^{\frac{2}{3}} e^{5\pi\iota/3},$$
therefore, since
$$\frac{\partial}{\partial\xi}\{\xi^{\frac{2}{3}}\mathrm{K}_{\frac{2}{3}}(\xi e^{-\pi\iota})\} = \xi^{\frac{2}{3}}\mathrm{K}_{\frac{1}{3}}(\xi e^{-\pi\iota}), \qquad \frac{\partial}{\partial\xi}\{\xi^{\frac{1}{3}}\mathrm{K}_{\frac{1}{3}}(\xi e^{-\pi\iota})\} = \xi^{\frac{1}{3}}\mathrm{K}_{\frac{2}{3}}(\xi e^{-\pi\iota}).$$
$$\frac{3}{2}y_k\xi\mathrm{K}_{\frac{1}{3}}(\xi e^{-\pi\iota}) + \frac{3}{4}y_k^2\xi^{\frac{1}{3}}\mathrm{K}_{\frac{1}{3}}(\xi e^{-\pi\iota}) - \frac{\kappa}{\iota\kappa'}3^{-\frac{1}{3}}z^{\frac{1}{3}}\xi^{\frac{1}{3}}\mathrm{K}_{\frac{1}{3}}(\xi e^{-\pi\iota}) = 0,$$
that is
$$\frac{3}{2}y_k\xi^{\frac{2}{3}} + \frac{3}{4}y_k^2 - \frac{\kappa}{\iota\kappa'}3^{-\frac{1}{3}}z^{\frac{1}{3}} = 0.$$

Now
$$\kappa^2/\kappa'^2 = \iota\sigma/2\lambda\mathrm{V},$$
therefore
$$\kappa/\kappa' = \eta e^{\frac{1}{4}\pi\iota},$$
where
$$\eta = (\sigma/2\lambda\mathrm{V})^{\frac{1}{2}},$$
whence
$$\tfrac{3}{2}y_k\xi^{\frac{2}{3}} + \tfrac{3}{4}y_k^2 = 3^{-\frac{1}{3}}z^{\frac{1}{3}}\eta e^{-\frac{1}{4}\pi\iota},$$
and
$$y_k = 2\,3^{-\frac{4}{3}}\xi^{-\frac{2}{3}}z^{\frac{1}{3}}\eta e^{-\frac{1}{4}\pi\iota} - 2\,3^{-\frac{8}{3}}\xi^{-2}z^{\frac{2}{3}}\eta^2 e^{-\frac{1}{2}\pi\iota},$$
therefore
$$(3x'_k)^{\frac{2}{3}} = (3x_k)^{\frac{2}{3}} + 2\eta z^{\frac{1}{3}}(3x_k)^{-\frac{2}{3}}e^{5\pi\iota/12} + 2\eta^2 z^{\frac{2}{3}}(3x_k)^{-2}e^{-\frac{1}{6}\pi\iota}. \tag{17}$$

Again
$$\frac{\xi^{\frac{1}{3}}\mathrm{K}_{\frac{1}{3}}(e^{-\pi\iota})}{\xi^{\frac{2}{3}}\mathrm{K}_{\frac{2}{3}}(\xi e^{-\pi\iota}) - \iota\kappa/\kappa' \,.\, 3^{-\frac{1}{3}}z^{\frac{1}{3}}\xi^{\frac{1}{3}}\mathrm{K}_{\frac{1}{3}}(\xi e^{-\pi\iota})} = \Sigma\frac{\mathrm{A}_k}{\xi^{\frac{2}{3}} - x'^{\frac{2}{3}}_k e^{5\pi\iota/3}},$$
where
$$\mathrm{A}_k = \frac{\xi^{\frac{1}{3}}\mathrm{K}_{\frac{1}{3}}(\xi e^{-\pi\iota})}{\frac{3}{2}\xi^{\frac{1}{3}}\frac{\partial}{\partial\xi}\{\xi^{\frac{2}{3}}\mathrm{K}_{\frac{2}{3}}(\xi e^{-\pi\iota})\} - \frac{3}{2}\frac{\kappa}{\iota\kappa'}3^{-\frac{1}{3}}z^{\frac{1}{3}}\xi^{\frac{1}{3}}\frac{\partial}{\partial\xi}\{\xi^{\frac{1}{3}}\mathrm{K}_{\frac{1}{3}}(\xi e^{-\pi\iota})\}},$$
when
$$\xi^{\frac{2}{3}} = x'^{\frac{2}{3}}_k e^{5\pi\iota/3},$$
that is
$$\mathrm{A}_k = \frac{\xi^{\frac{1}{3}}\mathrm{K}_{\frac{1}{3}}(\xi e^{-\pi\iota})}{\frac{3}{2}\xi\mathrm{K}_{\frac{1}{3}}(\xi e^{-\pi\iota}) - \frac{3}{2}\frac{\kappa}{\iota\kappa'}3^{-\frac{1}{3}}z^{\frac{1}{3}}\xi^{\frac{2}{3}}\mathrm{K}_{\frac{2}{3}}(\xi e^{-\pi\iota})},$$
or
$$\mathrm{A}_k = \frac{2}{3}\frac{1}{\xi^{\frac{2}{3}} - 3^{-\frac{1}{3}}z^{\frac{1}{3}}\frac{\kappa}{\iota\kappa'}\xi^{\frac{2}{3}}\mathrm{K}_{\frac{2}{3}}(\xi e^{-\pi\iota})/\xi^{\frac{1}{3}}\mathrm{K}_{\frac{1}{3}}(\xi e^{-\pi\iota})}.$$

Now, to the order of approximation required,
$$\xi^{\frac{2}{3}}\mathrm{K}_{\frac{2}{3}}(\xi e^{-\pi\iota})/\xi^{\frac{1}{3}}\mathrm{K}_{\frac{1}{3}}(e^{-\pi\iota}) = \frac{3}{2}y_k\xi^{\frac{1}{3}}\frac{\partial}{\partial\xi}\{\xi^{\frac{2}{3}}\mathrm{K}_{\frac{2}{3}}(\xi e^{-\pi\iota})\}/\xi^{\frac{1}{3}}\mathrm{K}_{\frac{1}{3}}(\xi e^{-\pi\iota}),$$
or
$$\xi^{\frac{2}{3}}\mathrm{K}_{\frac{2}{3}}(\xi e^{-\pi\iota})/\xi^{\frac{1}{3}}\mathrm{K}_{\frac{1}{3}}(\xi e^{-\pi\iota}) = 3^{-\frac{1}{3}}z^{\frac{1}{3}}\eta e^{-\frac{1}{4}\pi\iota},$$
and therefore
$$\mathrm{A}_k = \frac{2}{3}\frac{1}{\xi^{\frac{2}{3}} - 3^{-\frac{2}{3}}z^{\frac{2}{3}}\eta^2 e^{-\frac{1}{2}\pi\iota}},$$
that is
$$\mathrm{A}_k = \frac{2}{3^{\frac{1}{3}}e^{\frac{5}{3}\pi\iota}}\frac{1}{(3x'_k)^{\frac{2}{3}} - \eta^2 z^{\frac{2}{3}}e^{-\frac{1}{6}\pi\iota}},$$
or
$$\mathrm{A}_k = \frac{2}{3^{\frac{1}{3}}e^{\frac{5}{3}\pi\iota}}\frac{1}{(3x_k)^{\frac{2}{3}} + 2\eta z^{\frac{1}{3}}(3x_k)^{-\frac{2}{3}}e^{\frac{5}{12}\pi\iota} + \eta^2 z^{\frac{2}{3}}e^{-\frac{1}{6}\pi\iota}(2(3x_k)^{-2} - 1)}. \tag{18}$$

Again, substituting for $P_n(\mu)$ the appropriate approximation

$$P_n(\mu) = J_0\{(2n+1)\sin\tfrac{1}{2}\theta\},$$

the series for ψ becomes

$$-2\,.\,3^{-\frac{1}{3}}\pi^{-\frac{1}{2}}\kappa z^{-\frac{5}{3}}(\sin\tfrac{1}{2}\theta)^{\frac{1}{2}}\cos^2\tfrac{1}{2}\theta\,\Sigma(n+\tfrac{1}{2})^{\frac{3}{2}}\left[e^{(2n+1)\sin\frac{1}{2}\theta-\frac{3}{4}\pi\iota}+e^{-(2n+1)\sin\frac{1}{2}\theta+\frac{3}{4}\pi\iota}\right]$$

$$.\sum_{k=0}\frac{A_k}{\xi^{\frac{2}{3}}-x'_k{}^{\frac{2}{3}}e^{\frac{5}{3}\pi\iota}},$$

that is since $\quad n+\frac{1}{2} = 2+3\,.\,6^{-\frac{1}{3}}z^{\frac{1}{3}}\zeta,$

the principal part $\bar{\psi}$ of ψ is given by

$$\bar{\psi} = -2\pi^{-\frac{1}{2}}\kappa(3z)^{-\frac{1}{3}}(z^{\frac{1}{3}}\sin\tfrac{1}{2}\theta)^{\frac{1}{2}}\cos^2\tfrac{1}{2}\theta\left[e^{2\iota z\sin\frac{1}{2}\theta-\frac{3}{4}\pi\iota}S_1+e^{-2\iota z\sin\frac{1}{2}\theta+\frac{3}{4}\pi\iota}S_2\right],$$

where

$$S_1 = \Sigma\sum_{k=0}e^{6^{\frac{2}{3}}z^{\frac{1}{3}}\zeta\iota\sin\frac{1}{2}\theta}\frac{A_k}{\xi^{\frac{2}{3}}-x'_k{}^{\frac{2}{3}}e^{\frac{5}{3}\pi\iota}},\qquad S_2 = \Sigma\sum_{k=0}e^{-6^{\frac{2}{3}}z^{\frac{1}{3}}\zeta\iota\sin\frac{1}{2}\theta}\frac{A_k}{\xi^{\frac{2}{3}}-x'_k{}^{\frac{2}{3}}e^{\frac{5}{3}\pi\iota}},$$

therefore the principal part of S_1 vanishes, and the principal part of S_2 is*

$$-3^{\frac{2}{3}}\pi\iota z^{\frac{1}{3}}\sum_{k=0}e^{-(3x'_k)^{\frac{2}{3}}z^{\frac{1}{3}}\sin\frac{1}{2}\theta e^{\frac{1}{6}\pi\iota}}A_k,$$

that is

$$-3^{\frac{1}{3}}2\pi\iota z^{\frac{1}{3}}e^{-\frac{5}{3}\pi\iota}\sum_{k=0}^{\infty}e^{-(3x'_k)^{\frac{2}{3}}z^{\frac{1}{3}}\sin\frac{1}{2}\theta^{\frac{1}{6}\pi\iota}}/\{(3x'_k)^{\frac{2}{3}}-\eta^2z^{\frac{2}{3}}e^{-\frac{1}{6}\pi\iota}\},$$

and therefore

$$\bar{\psi} = -4\pi^{\frac{1}{2}}\iota\kappa(z^{\frac{1}{3}}\sin\tfrac{1}{2}\theta)^{\frac{1}{2}}\cos^2\tfrac{1}{2}\theta e^{-2\iota z\sin\frac{1}{2}\theta+\frac{1}{12}\pi\iota}\sum_{k=0}^{\infty}\frac{e^{-(3x'_k)^{\frac{2}{3}}z^{\frac{1}{3}}\sin\frac{1}{2}\theta)e^{\frac{1}{6}\pi\iota}}}{(3x'_k)^{\frac{2}{3}}-\eta^2z^{\frac{2}{3}}e^{-\frac{1}{6}\pi\iota}},\qquad(19)$$

neglecting powers of $\eta z^{\frac{1}{3}}$ higher than the second.

Comparing this result with the value of $\bar{\psi}$ when η vanishes, that is, with the value of $\bar{\psi}$ that corresponds to perfect conduction, it appears that the first term of the series is the one that is most affected, and at distances from the oscillator for which the first term gives a sufficient approximation to the value of $\bar{\psi}$ the effect of imperfect conduction is to multiply the value of $\bar{\psi}$ corresponding to perfect conduction by the factor

$$\frac{e^{-[2\eta z^{\frac{1}{3}}(3x_0)^{-\frac{2}{3}}e^{\frac{7}{12}\pi\iota}+2\eta^2z^{\frac{2}{3}}(3x_0)^{-2}]z^{\frac{1}{3}}\sin\frac{1}{2}\theta}}{1+2\eta z^{\frac{1}{3}}(3x_0)^{-\frac{4}{3}}e^{\frac{5}{12}\pi\iota}+\eta^2z^{\frac{2}{3}}(3x_0)^{-\frac{2}{3}}(2(3x_0)^{-2}-1)e^{-\frac{1}{6}\pi\iota}},$$

Hence, if A_ρ denotes the ratio of the amplitude of the magnetic force when the conduction is imperfect to the amplitude of the magnetic force when the conduction is perfect, A_ρ is given by

$$A_\rho = \frac{e^{[2\eta z^{\frac{1}{3}}(3x_0)^{-\frac{2}{3}}\sin\frac{1}{12}\pi-2\eta^2z^{\frac{2}{3}}(3x_0)^{-2}]z^{\frac{1}{3}}2^{\frac{1}{3}}\sin\frac{1}{2}\theta}}{\left|1+2\eta z^{\frac{1}{3}}(3x_0)^{-\frac{4}{3}}e^{\frac{5}{12}\pi\iota}+\eta^2z^{\frac{2}{3}}(3x_0)^{-2}\{2(3x_0)^{-2}-1\}e^{-\frac{1}{6}\pi\iota}\right|},\qquad(20)$$

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* 'Roy. Soc. Proc.,' vol. 90, 1914.

or

$$A_\rho = [1 - 2\eta z^{\frac{1}{3}}(3x_0)^{-\frac{4}{3}} \sin \tfrac{1}{12}\pi + \eta^2 z^{\frac{2}{3}}(3x_0)^{-\frac{8}{3}} - \eta^2 z^{\frac{2}{3}}(3x_0)^{-\frac{2}{3}}\{5(3x_0)^{-2} - 1\} \cos \tfrac{1}{6}\pi]$$

$$\times e^{[2\eta z^{\frac{1}{3}}(3x_0)^{-\frac{2}{3}} \sin \frac{1}{12}\pi - 2\eta^2 z^{\frac{2}{3}}(3x_0)^{-2}] z^{\frac{1}{3}} \sin \frac{1}{2}\theta}.$$

It follows that, when $\eta z^{\frac{1}{3}}$ is a small quantity, A_ρ increases with the angular distance θ from the oscillator and diminishes as the wave-length increases. It should be observed that the effect of the terms of the second order is to diminish the factor A_ρ. For a wave-length of 5 kilom. the value of $\eta z^{\frac{1}{3}}$ is 0·0365 and at angular distances of 6°, 9°, 12°, 15°, 18° from the oscillator the values of A_ρ neglecting the terms involving $\eta^2 z^{\frac{2}{3}}$ are 1·005, 1·011, 1·017, 1·023, 1·029.

These results, with the exception of the value corresponding to 15°, are in agreement with the results obtained from Love's* formula (51) when $(k/m)^2$ is neglected. At a distance from the oscillator corresponding to $\theta = 36°$, that is at a distance of 4000 kilom., the value of A_ρ is to the same order 1·064. When the terms of the second order are retained the formula (20) gives the values at 6°, 9°, 12°, 15°, 18°, 36° for A_ρ 1·004, 1·010, 1·015, 1·021, 1·027, 1·061, and the effect of imperfect conduction for a wave-length of 5 kilom. is to increase the amplitude at distances not greater than 4000 kilom. by not more than 6½ per cent. To indicate the effect of diminishing the wave-length the results for a wave-length of 2 kilom. have been calculated, and the corresponding values of A_ρ at the same distances are 1·020, 1·038, 1·056, 1·073, 1·091, 1·195 when the terms of the second order are neglected, and 1·016, 1·032, 1·049, 1·065, 1·082, 1·186 when the terms of the second order are retained.

No direct determination of the value of σ when waves are incident on sea-water appears to have been made; it is unlikely that σ has the same value for all wave-lengths, and it has not been established that σ has the same value as when a current of the same frequency is passed through the water. The value of σ for a given wave-length could be found from a knowledge of the electric disturbance at different depths when waves of the given length are passing over the surface. If it can be assumed that for waves of great length the value 10^{11} used in the calculation is of the right order, it follows that the effect of the imperfect conduction of the sea on the transmission of waves of such length over its surface is practically negligible.

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* 'Phil. Trans.,' A, vol. 215, p. 122 (1915).